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A UNIFIED FINITE ELEMENT METHOD FOR DETERMINING WEIGHT FUNCTIONS IN TWO AND THREE DIMENSIONS

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Abstract—A variational principle is introduced for determining *singular* fields in finite bodies with unbounded strain energy. A specialized form of this variational principle is employed to determine the Bueckner–Rice weight functions in two and three dimensions. The cracked body considered is finite, linear elastic and homogeneous but arbitrary anisotropy is permitted. It contains an arbitrary configuration of planar cracks with continuously turning crack fronts. A finite element implementation of the variational principle is given and this leads to a unified approach in the numerical computations of weight functions for all three fracture modes. The finite element method presented is simple and yet accurate and it can be incorporated into standard finite element programs. Numerical examples are given for two-dimensional crack geometries in isotropic bodies to illustrate the versatility and accuracy of the method.

1. INTRODUCTION

The weight function theory for calculating stress intensity factors in a linear elastic cracked body was introduced by Bueckner[1] in 1970. It leads to the representation

$$K_\alpha = \int_{\text{area}} \bar{\mathbf{T}} \cdot \mathbf{h}_\alpha \, dA + \int_{\text{volume}} \mathbf{F} \cdot \mathbf{h}_\alpha \, dV \tag{1}$$

of the stress intensity factors K_α for a cracked body under arbitrary combinations of Mode I and Mode II plane strain deformation at the crack tip. Here $\bar{\mathbf{T}}$ is the prescribed surface traction, $\mathbf{F}$ the body force per unit volume, $\mathbf{h}_\alpha$ are the weight functions and α takes on values 1 and 2, with each value denoting Mode I and Mode II deformation, respectively. Since Mode III deformations for two-dimensional crack configurations are decoupled from the other two modes, we can include this mode of deformation in eqn (1) by adding the value of 3 to the subscript α in eqn (1). The cracked body can consist of an arbitrary number of cracks and the vector fields $\mathbf{h}_\alpha(\mathbf{x})$ are the weight functions associated with a specific crack tip in the body at which the stress intensity factors K_α are to be determined.

A unique feature of these weight functions is that they are universal functions for the given crack configuration and body geometry. Once the weight functions associated with a specific crack tip in a given body are known, the stress intensity factors for that crack tip can be computed directly for any surface and body force loading by using eqn (1). The weight functions can be identified as displacements which obey all the equations of linear elasticity except one aspect which distinguishes them from some other displacement fields. The weight functions associated with a particular crack tip give rise to unbounded energy in any finite area surrounding that crack tip. Thus, their existence is outside the class of displacement fields corresponding to bounded energy upon which proof of uniqueness in linear elasticity theory is based. The stress fields generated by the weight functions are self-equilibrating; they have no body forces; and they produce zero tractions on the crack faces as well as the external boundary of the cracked body. The field of displacements, strains and stresses are also referred to as a *fundamental field*[1].

Within the framework of Muskhelishvili's theory Bueckner introduced fundamental field *per se*. He applied Betti's theorem of reciprocity in order to derive eqn (1) for the case

$\mathbf{F} \equiv 0$. This assumption as well as conditions of regularity on the distribution of $\bar{\mathbf{T}}$ on the boundary (crack faces included) make it possible to use the analytic function theory throughout the derivation. In the application of Betti's theorem a neighborhood of the crack tip is excluded and thereafter shrunk into the tip. It is noted that the same technique can be used to derive eqn (1) in its general form under suitable assumptions of regularity on the distribution of $\mathbf{F}$.

Through the concept of energetic forces and crack tip motions, Rice[2] has given a different method for determining the Mode I weight functions. He noted that if the stress intensity factor, $K_I(a)$, and displacement field, $\mathbf{u}_I(\mathbf{x}, a)$, for a cracked body under one symmetrical loading are known in their dependence on the crack tip position a, the Mode I weight function can be determined from

$$\mathbf{h}_I(\mathbf{x}, a) = \frac{M}{2K_I(a)} \frac{\partial \mathbf{u}_I(\mathbf{x}, a)}{\partial a} \tag{2}$$

where M is an appropriate elastic modulus; a is in the plane of the crack. The notion of energetic forces and their relations to defect motions is a very elegant concept and Rice[3] has demonstrated recently that a wide class of problems, such as: (i) Peach–Koehler force on a dislocation; (ii) interaction between dislocation and notch tip; (iii) interaction between dislocation and material interface; and (iv) structure of inelastic constitutive descriptions, could be investigated within the framework of energetic forces.

The weight function theory was further developed by Bueckner[4], whose investigation was independent of Rice's work[2]. Three specific methods for determining the weight functions in infinite as well as finite cracked bodies were given. They are: (i) computation of some regular fields (to be discussed in the subsequent section), (ii) parameter differentiation with respect to crack tip coordinate, and (iii) integral equations approach. It should be noted that this method of parameter differentiation is equivalent in form to Rice's method but it can be used to determine the weight functions in both Mode I and II. The weight functions of the configuration of collinear cracks were determined by Bueckner in Ref. [5]. Bueckner's approach for the weight functions in cracked bodies can also be used to determine generalized stress intensity factors for sharp notches[6].

The theoretical foundation of the two-dimensional weight functions are well laid out in Refs [1, 2, 4]; in particular: (i) the determination of weight functions under mixed boundary conditions (p. 264 in Ref. [4]); and (ii) the determination of weight functions under mixed mode deformations by the method of parameter differentiation (pp. 265–266 in Ref. [4]). However, it seems that this is not quite well appreciated and various generalizations and extensions of the two-dimensional weight function theory have been given, e.g. by Wu and Carlsson[7] and by Bortman and Banks-Sills[8].

The extension of the weight function theory to three-dimensional configurations was given independently by Rice in the Appendix of his article[2] through the consideration of variation in crack front locations and the associated energetic forces and by Bueckner[4] through the generalization of his two-dimensional weight function theory. Subsequent developments of the three-dimensional theory can be found in the series of articles by Bueckner[9–11] and Rice[12, 13].

The determination of weight functions for finite cracked bodies, using analytical techniques, is generally rather formidable and some form of numerical scheme is usually employed. There are two general types of numerical procedures for computing weight functions. The first type of procedure involves expressing the weight functions in terms of integral equations which are then solved by appropriate numerical techniques. Examples of this approach are given in Ref. [4]. The second type of technique is based on finite element methods and we shall focus on this class of methods.

Paris and McMeeking[14] and Paris *et al.*[15] have given a finite element procedure, which will be referred to as the PMT method, to compute directly the weight functions for two-dimensional crack configurations. As indicated above, the energy of deformation associated with the weight function in a finite area surrounding the crack tip is unbounded.

Thus, the standard finite element procedure which involves the minimization of the potential energy of the cracked body would fail. Paris and co-workers[14, 15] circumvented this difficulty by removing a cylinder of material centered at the crack tip, and solved an auxiliary boundary value problem for this new geometry. In the PMT method, the weight function for the given crack geometry in an infinite body is prescribed as displacement boundary conditions on the cylindrical surface while zero traction is maintained on the remaining part of the bounding surface to the body. In essence, the problem is regularized by removing the unboundedness of the strain energy near the crack tip. Hence, a standard finite element method, which is based on the principle of minimum potential energy, can be used to solve for the displacement field in this auxiliary problem. The resulting displacement solution is identified as the weight function. The accuracy of the solution obtained by the PMT method is quite adequate for engineering applications in general. But numerical experiments show that the accuracy of the solution is very sensitive to the radius of the cylinder of material being removed from the crack tip once this radius is smaller than some critical value. We shall address this aspect in a later section.

A different method for computing the weight function was given independently by Parks and Kamenetzky[16] and Vanderglas[17]. They adopted Rice's method[2] and computed the Mode I weight function according to eqn (2) through a finite element procedure which employs the virtual crack extension technique[18, 19]. This procedure is efficient for computing Mode I weight functions in general but it has two drawbacks. The first concerns the differencing used in certain parts of the finite element mesh for computing the stiffness derivative. All the disadvantages that go with numerical differentiation would apply to such computation. Secondly, eqn (2) involves the displacement derivatives with respect to the crack tip coordinates and there are limitations, as discussed in Refs [16, 17], associated with such a computation because of the special perturbation technique used for some nodal points near the crack tip in simulating the virtual crack extension. Nevertheless, Parks and Kamenetzky[16] reported that the accuracy of the solution was comparable with the PMT method and they also outlined a generalization of their numerical procedure to the three-dimensional crack configurations by following Rice's[2] three-dimensional weight function theory development.

Most of the other finite element computations of the weight functions reported in the literature[20–22] are based on the implementation of Rice's method. Recent activity, for example[23, 24], has concentrated on the determination of mixed mode weight functions by generalizing the method given by Parks and Kamenetzky[16] and Vanderglas[17]. Such mixed mode numerical procedures, for example in Refs. [23, 24], would retain the same drawbacks as in the pure Mode I case commented upon earlier. Also, additional programming steps have to be incorporated into the already sophisticated finite element algorithm[16, 17]. It will be shown that the unified finite element method, to be introduced, will not have such complication.

In this paper, we introduce a variational principle for determining the singular fields in finite bodies with unbounded strain energy. We show how this variational principle can be specialized for three-dimensional weight functions in a finite, linear elastic and homogeneous body with arbitrary anisotropy, containing a single or a system of planar cracks. This principle will form the basis for implementing a unified element method for computing the weight functions under arbitrary combinations of all three fracture modes in both two and three dimensions. As discussed in later sections, the implementation of this finite element method is very straightforward and can be incorporated into any standard two- or three-dimensional linear elastic finite element programs. The synopsis of this paper is as follows. A variational principle developed in Appendix A will be specialized for the weight functions with mixed boundary conditions in Section 2. Section 3 discusses the finite element implementation of the variational principle. In Section 4, numerical examples will be given for the special case of computing two-dimensional weight functions in linear elastic isotropic solids. These are done to illustrate the simplicity and accuracy of this numerical method. The weight function calculations in three dimensions follow essentially the same strategy as in the two-dimensional case. The three-dimensional computations will be reported in a separate work[25].

2. A VARIATIONAL PRINCIPLE FOR DETERMINING THE WEIGHT FUNCTIONS

Consider a finite three-dimensional solid containing a system of planar cracks which have continuously turning crack fronts as depicted in Fig. 1(a). The solid is linear elastic and homogeneous but arbitrary anisotropy is permitted. A loading system which consists of surface forces, imposed boundary displacements and body forces is applied to the solid. At a generic point Q on the crack front of a particular planar crack, the loading system induces the three modes of stress intensities and we are interested in determining the weight functions, $\mathbf{h}_\alpha(\mathbf{x}; Q)$, for that generic point. We shall use a Cartesian coordinate system x_1, x_2, x_3 centered at Q where x_1 and x_3 are normal and tangential to the crack front, respectively, and x_2 is normal to the crack plane. Under mixed boundary conditions, the formulae given in eqn (1) for determining stress intensity factors can be modified to give

$$K_\alpha = \int_{\partial V_T} \bar{\mathbf{T}} \cdot \mathbf{h}_\alpha \, dA - \int_{\partial V_u} \mathbf{t}_\alpha \cdot \bar{\mathbf{U}} \, dA + \int_V \mathbf{F} \cdot \mathbf{h}_\alpha \, dV \tag{3}$$

where V is the volume of the body and $\partial V = \partial V_T \cup \partial V_u$ is the bounding surface of V. Here $\bar{\mathbf{T}}$ is the prescribed surface traction on ∂V_T, $\bar{\mathbf{U}}$ is the prescribed boundary displacement on ∂V_u, $\mathbf{F}$ is the prescribed body force in V and $\mathbf{t}_\alpha$ are the tractions generated by the weight functions $\mathbf{h}_\alpha$ on ∂V_u.

With reference to Fig. 1(b), let the generic point Q be isolated by a *suitably small* bounding surface from the rest of the body. We denote the part of the body enclosed by the bounding surface as region B and the remaining part of the body as region A. The bounding surface of region A is ∂A which consists of ∂A_T, ∂A_u (on which boundary tractions and displacements are imposed respectively by the loading system) and S_{int} (which separates regions A and B). The outward unit normal to ∂A is $\mathbf{n}^A$. In a similar vein, the bounding surface region B will be denoted as ∂B which is composed of ∂B_T, ∂B_u and S_{int} and the outward unit normal to ∂B will be denoted as $\mathbf{n}^B$.

The weight functions, $\mathbf{h}_\alpha(\mathbf{x}; Q)$, associated with the point Q and interpreted as displacements will have the following properties; they are zero on ∂V_u, the stress fields generated by $\mathbf{h}_\alpha(\mathbf{x}; Q)$ are self-equilibrating, they have no body forces and they produce

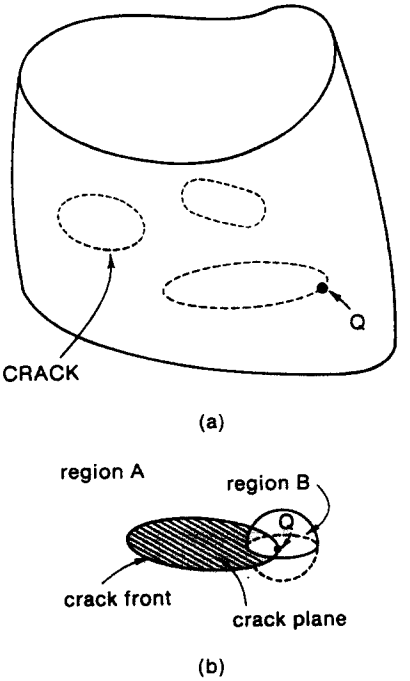

Fig. 1. (a) A three-dimensional solid containing a system of planar cracks which have continuously turning crack fronts; (b) the point Q on the crack front being enclosed by a bounding surface ∂B which separates the body into region B (inside ∂B) and region A (outside ∂B).

zero surface tractions on ∂V_T which includes all crack faces. We assume from here on that ∂V_u does not intersect the crack faces and ∂B_u is empty.† As points on the crack fronts, excluding point Q, are approached, $\mathbf{h}_\alpha(\mathbf{x};Q)$ behave like $O(d^\gamma)$, with $\gamma > 0$, where d is the distance measured from the specific point on the crack front at which the approach is taken. However, the asymptotic behavior of $\mathbf{h}_\alpha(\mathbf{x};Q)$ at point Q is very much different. As Q is approached, $\mathbf{h}_\alpha(\mathbf{x};Q)$ behave like $O(d^\lambda)$, with $\lambda < 0$.

In the works of Bueckner[1, 4, 9, 11] and Rice[2], the weight functions $\mathbf{h}_\alpha(\mathbf{x};Q)$ for a finite body are decomposed globally into some suitable singular displacements, $\mathbf{u}^s_\alpha(\mathbf{x};Q)$, and their supplementary regular displacements, $\mathbf{u}^r_\alpha(\mathbf{x};Q)$. The singular displacements $\mathbf{u}^s_\alpha(\mathbf{x};Q)$ have the same asymptotic behavior as $\mathbf{h}_\alpha(\mathbf{x};Q)$ at all points along the crack fronts; while the regular displacements behave like $O(d^\gamma)$, with $\gamma > 0$, as each point on the crack fronts is approached. Both $\mathbf{u}^s_\alpha(\mathbf{x};Q)$ and $\mathbf{u}^r_\alpha(\mathbf{x};Q)$ generate self-equilibrating stress fields, $\boldsymbol{\sigma}^s_\alpha$ and $\boldsymbol{\sigma}^r_\alpha$ respectively, which produce zero body forces and induce zero tractions on all crack faces. But, unlike the weight functions, the magnitudes of the singular displacements $\mathbf{u}^s_\alpha(\mathbf{x};Q)$ on ∂V_u and the tractions generated by them on the remaining parts of ∂V_T are nonzero. On the other hand, the magnitudes of the regular displacements $\mathbf{u}^r_\alpha(\mathbf{x};Q)$ on ∂V_u and the tractions on the same portions of ∂V_T generated by $\mathbf{u}^r_\alpha(\mathbf{x};Q)$ are exactly equal and opposite to those of the singular displacements $\mathbf{u}^s_\alpha(\mathbf{x};Q)$.

Sometimes one can interpret the singular displacements $\mathbf{u}^s_\alpha(\mathbf{x};Q)$ as the weight functions for the same crack configuration imbedded in an infinite as opposed to a finite body. The singular displacements $\mathbf{u}^s_\alpha(\mathbf{x};Q)$ then generate non-zero boundary displacements and tractions on the external boundary of the finite body which is under consideration. The weight functions for the finite body are constructed by employing the regular displacements $\mathbf{u}^r_\alpha(\mathbf{x};Q)$ to annihilate these unwanted boundary displacements and tractions.

To determine the weight functions in a finite body, we shall apply the variational principle developed in Appendix A for singular fields in finite bodies to our problem on hand. In order to utilize this variational principle appropriate approximations of $\mathbf{h}_\alpha(\mathbf{x};Q)$ have to be found in region B such that the differences between $\mathbf{h}_\alpha(\mathbf{x};Q)$ and the approximations correspond to *regular* displacements. To this end, we first observe that the weight functions for two different crack configurations and body geometries differ from one another merely by some regular displacement fields. This can be illustrated by considering the weight functions for (i) a semi-infinite crack in an infinite space; and (ii) an internal crack of finite size in an infinite body, both deforming in plane strain. When the weight functions for the semi-infinite crack are applied to the internal crack configuration, they yield the correct singularity at one tip (say Q) of the internal crack but undue openings are produced on some portions of the crack plane in the internally cracked body. These openings can be closed up by applying appropriate surface tractions which would induce stress intensities at the crack tip Q. The elastic fields induced by these surface tractions are necessarily regular and by the principle of superposition the weight functions for the internal crack can be obtained by adding to the weight functions of the semi-infinite crack some appropriate regular displacements.

Let the weight functions $\mathbf{h}_\alpha(\mathbf{x};Q)$ in region B be approximated by the singular displacements $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x};Q)$ associated with a different but *simple* crack geometry in an infinite body, e.g. a semi-infinite crack in two dimensions or a penny-shaped crack in three dimensions. The singular stress fields $\tilde{\boldsymbol{\sigma}}^s_\alpha$ generated by $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x};Q)$ are in equilibrium with zero body forces in B and they induce zero tractions on the crack faces inside B. We shall call $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x};Q)$ the modified singular displacements which can be constructed independently no matter how complicated the geometry of the actual system of cracks is. The differences between the weight functions and the modified singular displacements, $\mathbf{h}_\alpha - \tilde{\mathbf{u}}^s_\alpha$, in region B will be referred to as the modified regular displacements, $\tilde{\mathbf{u}}^r_\alpha(\mathbf{x};Q)$, which are not known *a priori*. The modified regular stress fields, $\tilde{\boldsymbol{\sigma}}^r_\alpha$, of $\tilde{\mathbf{u}}^r_\alpha$ are in equilibrium with zero body forces in B and generate zero tractions on all crack faces inside B.

Following Appendix A, the displacements $\mathbf{h}_\alpha$ in A and $\tilde{\mathbf{u}}^r_\alpha$ in B are kinematically admissible if:

† These restrictions can be relaxed to include imposed displacements on crack faces by the loading system.

(i) they satisfy the strain–displacement relations in their respective regions;
(ii) $\mathbf{h}_\alpha - \tilde{\mathbf{u}}^r_\alpha = \tilde{\mathbf{u}}^s_\alpha$ on S_{int};
(iii) and $\mathbf{h}_\alpha = 0$ on ∂V_u.

With these definitions for the modified singular and regular fields in B, the functional H developed in Appendix A can be applied to the kinematically admissible fields $\mathbf{h}_\alpha$ in A and $\tilde{\mathbf{u}}^r_\alpha$ in B. Then H takes the form

$$H[\mathbf{h}_\alpha, \tilde{\mathbf{u}}^r_\alpha] = \int_A w(\boldsymbol{\varepsilon}_\alpha)\, dV + \int_B w(\tilde{\boldsymbol{\varepsilon}}^r_\alpha)\, dV - \int_{S_{int}} (-\tilde{\boldsymbol{\sigma}}^s_\alpha \mathbf{n}^B)\cdot \tilde{\mathbf{u}}^r_\alpha\, dA \tag{4}$$

where we have set $\mathbf{f}$, $\mathbf{f}^r$, $\mathbf{T}$ and $\mathbf{T}^r$ in eqn (A1) to zero for this specialization. Here $\boldsymbol{\varepsilon}_\alpha$ and $\tilde{\boldsymbol{\varepsilon}}^r_\alpha$ are the strains corresponding to $\mathbf{h}_\alpha$ and $\tilde{\mathbf{u}}^r_\alpha$ respectively and w is the linear elastic strain energy density. Direct notation for tensorial quantities are used and no summation is implied on repeated Greek subscripts. As shown in Appendix A, the functional H given in eqn (4) is bounded. It is a functional of the kinematically admissible fields $\mathbf{h}_\alpha$ in region A and $\tilde{\mathbf{u}}^r_\alpha$ in region B and the modified singular displacements $\tilde{\mathbf{u}}^s_\alpha$ and stresses $\tilde{\boldsymbol{\sigma}}^s_\alpha$ are considered to be given. The functional H attains a minimum for the true fields and the variational statement

$$\delta H[\mathbf{h}_\alpha, \tilde{\mathbf{u}}^r_\alpha] = 0 \tag{5}$$

where δH is the first variation of H, forms the basis for a finite element method which will be given in the following section. We now examine this minimum principle in more detail.

For cases where global singular displacements $\mathbf{u}^s_\alpha(\mathbf{x}; Q)$ for a given crack geometry are known, we can choose the modified singular displacements $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x}; Q)$ to be the same as $\mathbf{u}^s_\alpha(\mathbf{x}; Q)$. With such a choice, the functional H in eqn (4) becomes

$$H[\mathbf{h}_\alpha, \mathbf{u}^r_\alpha] = \int_A w(\boldsymbol{\varepsilon}_\alpha)\, dV + \int_B w(\boldsymbol{\varepsilon}^r_\alpha)\, dV - \int_{S_{int}} (-\boldsymbol{\sigma}^s_\alpha \mathbf{n}^B)\cdot \mathbf{u}^r_\alpha\, dA. \tag{6}$$

The strain energy term associated with $\boldsymbol{\varepsilon}_\alpha$ in region A can be written as

$$\int_A w(\boldsymbol{\varepsilon}_\alpha)\, dV = \int_A w(\boldsymbol{\varepsilon}^s_\alpha)\, dV + \int_A w(\boldsymbol{\varepsilon}^r_\alpha)\, dV + \int_{S_{int}+\partial A_T} (\boldsymbol{\sigma}^s_\alpha \mathbf{n}^A)\cdot \mathbf{u}^r_\alpha\, dA.$$

Here we have assumed for convenience that the imposed loading system corresponds to the traction type when the stress intensity factors are determined through the weight functions. In addition, we have used the equilibrium of $\boldsymbol{\sigma}^s_\alpha$ in A in writing this expression. Of course, the tractions $\boldsymbol{\sigma}^s_\alpha \mathbf{n}^A$ on the parts of ∂A_T which correspond to the crack faces are zero. Substituting the last expression into eqn (6) gives

$$H[\mathbf{u}^r_\alpha] = \int_V w(\boldsymbol{\varepsilon}^r_\alpha)\, dV - \int_{\partial A_T} (-\boldsymbol{\sigma}^s_\alpha \mathbf{n}^A)\cdot \mathbf{u}^r_\alpha\, dA + \int_A w(\boldsymbol{\varepsilon}^s_\alpha)\, dV \tag{7}$$

where volume V is the sum of A and B and the condition $\mathbf{n}^A = -\mathbf{n}^B$ on S_{int} is used. The functional H in eqn (7) is now a functional of $\mathbf{u}^r_\alpha(\mathbf{x}; Q)$ alone in both A and B. Also, H has bounded energy because the strain energy associated with $\boldsymbol{\varepsilon}^s_\alpha$ in A is finite. Thus the minimum principle specializes to the procedure given by Bueckner[1, 4, 9, 11] for determining the weight functions in a finite body through the calculation of the regular displacements $\mathbf{u}^r_\alpha(\mathbf{x}; Q)$.

On the other hand, if $\tilde{\mathbf{u}}^r_\alpha(\mathbf{x}; Q)$ in B is chosen to be zero as an approximation, H in eqn (4) becomes

$$H[\mathbf{h}_\alpha] = \int_A w(\varepsilon_\alpha)\, \mathrm{d}V$$

with $\mathbf{h}_\alpha$ being set equal to $\mathbf{u}^s_\alpha$ on the internal boundary S_{int}. This specialization of H can be interpreted as to removing region B from consideration and to solving for the weight functions $\mathbf{h}_\alpha$ in region A alone. This is equivalent to the PMT procedure.

However, the proposed finite element method which is based on the present variational principle has some advantages over Bueckner's method and the PMT method. As shown earlier, when the singular displacements $\mathbf{u}^s_\alpha(\mathbf{x}; Q)$ are known for a given crack configuration in an infinite body, Bueckner's method of determining the regular displacements and the present method are equivalent and they both give numerical solutions of comparable accuracy. But unlike the PMT method, the accuracy of the solutions obtained by the present method does not depend on the smallness of the size of region B chosen and this will be illustrated in examples to follow. For cases where the singular displacements $\mathbf{u}^s_\alpha(\mathbf{x}; Q)$ in an infinite body for the given crack geometry are not known, the global decomposition of the weight functions into singular and regular displacements cannot be carried out. Bueckner's method of determining the regular displacements $\mathbf{u}^r_\alpha(\mathbf{x}; Q)$ will fail since the tractions $\boldsymbol{\sigma}^s_\alpha \mathbf{n}$ are not known *a priori* on all parts of ∂A_T. However, in the present method, only a *local* decomposition of the weight functions is required. Thus we only need to know the modified singular tractions $\tilde{\boldsymbol{\sigma}}^s_\alpha \mathbf{n}$ and modified singular displacements $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x}; Q)$ on the internal surface S_{int}. The construction of the modified singular displacements $\tilde{\mathbf{u}}^s_\alpha(\mathbf{x}; Q)$ can be done, quite trivially in two dimensions and are available in three dimensions from Bueckner's[9–11] and Rice's[12, 13] work, irrespective of the complexity of the actual crack configuration in the body under study. This exceptional feature of the minimum principle allows us to implement a unified finite element method for calculating weight functions in both two and three dimensions.

3. FINITE ELEMENT IMPLEMENTATION OF THE VARIATIONAL PRINCIPLE

The implementation of the variational statement (5) within the context of standard displacement finite element method is very straightforward. The two regions, A and B, are discretized into finite elements with the nodal variables in A interpreted as the weight functions $\mathbf{h}_\alpha$ and those in B as the modified regular displacements $\tilde{\mathbf{u}}^r_\alpha$. Biquadratic and triquadratic finite elements can be used respectively for two-dimensional and three-dimensional computations.

In order that the nodal displacements for $\mathbf{h}_\alpha$ and $\tilde{\mathbf{u}}^r_\alpha$ are kinematically admissible, the condition

$$\mathbf{h}_\alpha = \tilde{\mathbf{u}}^s_\alpha + \tilde{\mathbf{u}}^r_\alpha$$

is enforced on S_{int} as follows. Consider a finite element in region A which borders the surface S_{int} as depicted in Fig. 2. The strain energy for this finite element, W_e, is

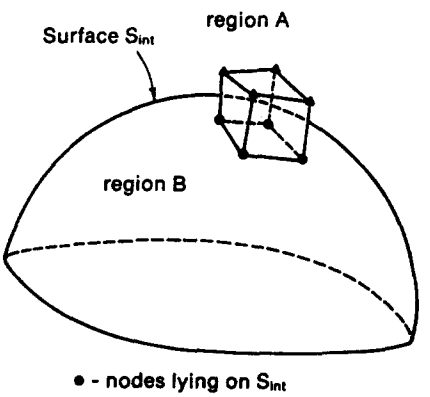

Fig. 2. The partitioning of the nodal vector $\{U\}$ into $\{U_1\}$ and $\{U_2\}$ for a finite element in region A which lies next to the internal surface S_{int}.

$$W_e = \int_{\text{element}} w(\varepsilon_\alpha)\, dV = \frac{1}{2}\int_{\text{element}} \varepsilon_\alpha \cdot \mathbf{C}\varepsilon_\alpha\, dV$$

where $\mathbf{C}$ is the elasticity tensor. Following the standard finite element discretization procedure, we can express $\mathbf{h}_\alpha$ in terms of the nodal weight function vector $\{U\}$ for this finite element through the shape functions. Suppose this nodal vector $\{U\}$ is partitioned such that $\{U_1\}$ corresponds to the nodal weight functions for those nodes lying on the surface S_{int} and $\{U_2\}$ is the remaining part of $\{U\}$. Then, the strain energy for this element can be expressed in terms of matrix notation as

$$W_e = \tfrac{1}{2}\{\{U_1\}^T \mid \{U_2\}^T\}\begin{bmatrix}[K_{11}] & [K_{12}]\\ [K_{21}] & [K_{22}]\end{bmatrix}\begin{Bmatrix}\{U_1\}\\ \{U_2\}\end{Bmatrix}$$

where the $[K_{ij}]$'s are the appropriate partitions of the finite element stiffness matrix. Now writing

$$\{U_1\} = \{\tilde{U}^s\} + \{\tilde{U}^r\}$$

in which $\{\tilde{U}^s\}$ and $\{\tilde{U}^r\}$ are the respective nodal values of the modified singular displacements $\tilde{\mathbf{u}}^s_\alpha$ and the modified regular displacements $\tilde{\mathbf{u}}^r_\alpha$, the strain energy W_e becomes

$$\begin{aligned} W_e = {} & \tfrac{1}{2}\{\{\tilde{U}^r\}^T \mid \{U_2\}^T\}\begin{bmatrix}[K_{11}][K_{12}]\\ [K_{21}][K_{22}]\end{bmatrix}\begin{Bmatrix}\{\tilde{U}^r\}\\ \{U_2\}\end{Bmatrix} \\ & + \tfrac{1}{2}\{\{\tilde{U}^r\}^T \mid \{U_2\}^T\}\begin{Bmatrix}[K_{11}]\{\tilde{U}^s\}\\ [K_{21}]\{\tilde{U}^s\}\end{Bmatrix} \\ & + \tfrac{1}{2}\{\{\tilde{U}^s\}^T[K_{11}] \mid \{\tilde{U}^s\}^T[K_{12}]\}\begin{Bmatrix}\{\tilde{U}^r\}\\ \{U_2\}\end{Bmatrix} + \tfrac{1}{2}\{\tilde{U}^s\}^T[K_{11}]\{\tilde{U}^s\}. \end{aligned}$$

Note that the modified singular displacements $\tilde{\mathbf{u}}^s_\alpha$ are considered to be known, so the last term of W_e does not contribute to the finite element equilibrium equations when the discretized functional H is minimized with respect to the nodal unknowns $\{U\}$ and $\{\tilde{U}^r\}$. The partitioning of the nodal vectors in the first term of W_e allows us to interpret those nodes lying on the surface S_{int} as the nodal values of the modified regular displacements $\{\tilde{U}^r\}$. Also, the second and third term of W_e act like a work term for this finite element, produced by an equivalent nodal body force vector $\{B\}$ on the nodal displacements $\{\tilde{U}^r\}$ and $\{U_2\}$. This equivalent body force vector $\{B\}$ is

$$\{B\} = \begin{Bmatrix}-[K_{11}]\{\tilde{U}^s\}\\ -[K_{21}]\{\tilde{U}^s\}\end{Bmatrix}. \tag{8}$$

The terms corresponding to the work done on the surface S_{int} by the tractions $-\tilde{\boldsymbol{\sigma}}^s_\alpha \mathbf{n}^B$ in the expression for H in eqn (4) can also be discretized in the usual manner through the shape functions. This would lead to a load vector for the finite element equilibrium equations. Finally, the mixed boundary conditions on the external surface of the body can be enforced by standard finite element technique.

Thus, in the finite element implementation of the variational principle, we can discretize the regions and assemble the element stiffness matrices and load vectors as in the standard procedure. The nodal variables in region B, including those lying on the surface S_{int}, are interpreted as nodal values for the modified regular displacements $\tilde{\mathbf{u}}^r_\alpha$ and the remaining nodal variables would then represent the nodal values of the weight functions $\mathbf{h}_\alpha$. For those finite elements in region A lying next to the surface S_{int}, we would include a body force term $\{B\}$ given in eqn (8) to the finite element equilibrium equations. It is noted that the

equivalent body force vector $\{B\}$ can be computed in the finite element program without actually carrying out the partitioning. We can use the element stiffness matrix assembled in its standard form and perform its matrix multiplication with $\{\tilde{U}^s\}$ by using a permutation vector (an integer pointer).

Some additional observations are made on this finite element implementation. The weight functions $\mathbf{h}_\alpha$ are only singular at the point Q and they behave like $O(d^\gamma)$, with $\gamma > 0$, at all other points on the crack fronts. The modified regular displacements $\tilde{\mathbf{u}}_\alpha^r$ also behave like $O(d^\gamma)$ at every single point on the crack fronts. Since the value of γ for elastic sharp cracks is 1/2, we should use linear elastic singular elements, having a $1/\sqrt{d}$ stress singularity along all crack fronts in the body in order to achieve the best possible numerical accuracy. The quarter-point Lagrange elements given by Hussain *et al.*[26] were used in the two-dimensional problems reported in the next section. They are easy to use and their extension to three dimensions is straightforward.

4. EXAMPLES FOR TWO-DIMENSIONAL ISOTROPIC BODY

In this section, we shall specialize the finite element method to two dimensions by taking the length of the body to be infinite in one direction. Material isotropy is assumed so that we can illustrate the simplicity and generality of this finite element method by comparing the results with known solutions in two-dimensional weight function theory. In this specialization, the internal surface S_{int} will degenerate to a cylindrical one and the point Q will refer to a crack tip in the body.

We shall choose the singular displacements of a semi-infinite crack in an infinite body as the modified singular displacements $\tilde{\mathbf{u}}_\alpha^s$ in region B for all two-dimensional crack configurations. They are given in Appendix B for all three fracture modes for completeness.

4.1. *Single edge crack in a finite strip*

The first example that we consider is the problem of an edge crack in a finite strip deforming in plane strain. It is noted that the chosen modified singular displacements and the singular displacements for this cracked strip are the same. The crack length to strip width ratio a/b and strip height to width ratio L/b are 0.5 and 6, respectively. We would compute the Mode I weight function $\mathbf{h}_1$ for this cracked strip. Three different meshes were used and they are shown in Figs 3(a)–(c). We take region B to consist of the first two rings of finite elements nearest to the crack tip for all three meshes. The first ring of finite elements at the crack tip are the Lagrange quarter-point singular elements[26] and the remaining ones are biquadratic Lagrange elements. The sizes of the singular elements normalized with respect to the crack length for the three meshes shown in Figs 3(a)–(c) are 0.1886, 0.0926, and 0.0459, respectively. The Mode I weight function $\mathbf{h}_1$ was computed for each mesh by prescribing the Mode I modified singular displacement $\tilde{\mathbf{u}}_1^s$ listed in Appendix B in the manner as described in Section 3. The Mode I stress intensity factor K_1 for the case of

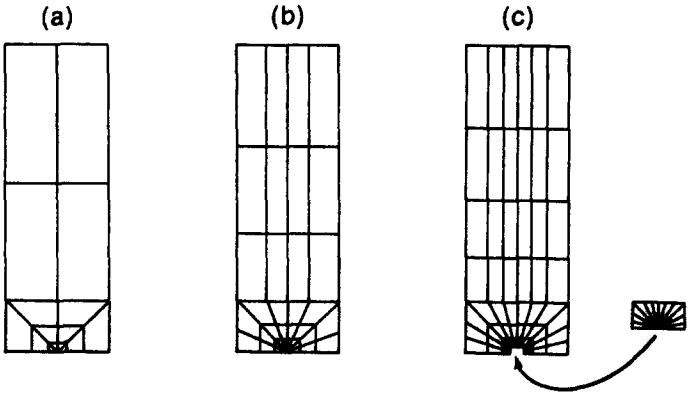

Fig. 3. Finite element mesh for the symmetrical half of an edge-cracked strip with (a) 75 nodes; (b) 191 nodes; and (c) 355 nodes. The crack length to strip width ratio is 0.5.

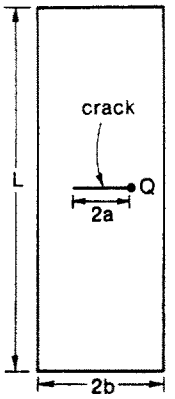

Fig. 4. A strip with an internal crack of length $2a$. Point Q is the crack tip for which the Mode I weight function is determined.

uniform tension σ was calculated according to eqn (3), using the computed weight function for each mesh. The resulting normalized stress intensity factors $K_1(\sigma\sqrt{(\pi a)})$ for the meshes in Figs 3(a)–(c) are 2.77605, 2.81656, and 2.82115, respectively. The result obtained from Tada *et al.*[27] is 2.82659. The finite element results compare very well with the result of Tada *et al.*[27] for all three meshes.

Using the mesh in Fig. 3(c), we next varied the size of region B so that it coincided in turn with the first one to four rings of finite elements nearest to the crack tip. The Mode I weight function $\mathbf{h}_1$ was computed again for the cracked strip with these four different areas for B and the corresponding values of K_1 induced by uniform tension were calculated accordingly. The computed values of $K_1/(\sigma\sqrt{(\pi a)})$ for the size of region B being 1, 2, 3, and 4 rings are 2.82862, 2.82321, 2.82191, and 2.82142, respectively. The differences in these normalized K_1 values are less than 0.3%. This shows that the accuracy of the computed weight functions are not sensitive to the size of region B chosen in the finite element mesh.

4.2. *Center crack in finite strip*

The second example that we consider corresponds to a center crack in a finite strip under plane strain conditions. In this case, the chosen modified singular displacements are not the same as the singular displacements for this center crack geometry. This example is chosen to illustrate that this finite element method will still give accurate weight functions in such cases. The finite element meshes used for the center-cracked strip were obtained by taking the mirror image of each mesh shown in Figs 3(a)–(c) with respect to the cracked side of the strip. The crack length is then taken to be $2a$. We computed the Mode I weight function $\mathbf{h}_1(\mathbf{x}; Q)$ associated with the crack tip at the right-hand side shown schematically in Fig. 4. The weight function $\mathbf{h}_1$ was used to determine the stress intensity factors corresponding to (i) uniform tensile loading σ applied to the top and bottom faces of the strip, and (ii) uniform pressure loading σ applied to the crack faces. The resulting normalized stress intensity factors $K_1/(\sigma\sqrt{(\pi a)})$ for the three meshes constructed from Figs 3(a)–(c) are; respectively: 1.17971, 1.18631, 1.18723 for loading (i); and 1.18169, 1.18683, 1.18762 for loading (ii). The normalized K_1 value from Tada *et al.*[27] is 1.18670. Again, excellent agreements are obtained. It should be noted that the K_1 values obtained by using the weight function on the crack face are as accurate as those calculated using the weight function on the top face of the strip.

4.3. *Single inclined edge crack in a finite strip*

The last example that we consider is a single inclined edge crack in a finite strip. Plane strain deformation is also assumed. Because the geometry of the strip is not symmetrical with respect to the crack plane, mixed mode stress intensities will be induced at the crack tip under in-plane loading. Even though we can separate the Mode I and II contributions from the mixed mode weight function in the conjugate parts of the body with respect to the crack plane[28], it might still be of interest to compute the Mode I and II weight

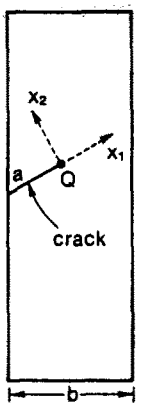

Fig. 5. The geometry of an inclined edge-cracked strip. The weight functions are computed with respect to the crack tip coordinate system shown.

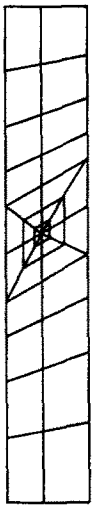

Fig. 6. Finite element mesh for the inclined edge-cracked strip. The crack length to strip width ratio is 0.5 and the angle between the crack plane and the vertical is 60°. Region B consists of the first ring of singular elements surrounding the crack tip. There are 183 nodes.

functions independently for such non-symmetrical geometry. This would be the case when *effective* body force loading is involved, for example in transformation strain problems[13], because we need to use the weight functions in all parts of the body, not just the conjugate parts, in order to calculate the stress intensity factor.

In the present finite element method, it is no more difficult to compute the Mode I and II weight functions for non-symmetrical crack geometries than the symmetrical ones. The *trick*[4], is to perform the computation with respect to the crack tip coordinate system shown in Fig. 5. With all quantities expressed in this coordinate system, the procedure for determining the Mode I and II weight functions is exactly the same as in the previous examples. For the computation of the Mode I weight function $\mathbf{h}_1$, we shall prescribe the Mode I modified singular displacement $\tilde{\mathbf{u}}_1^s$ as we have done before. Once the Mode I weight function $\mathbf{h}_1$ is obtained, K_1 can be calculated by using eqn (3), with the understanding that such calculation is done using the crack tip coordinate system. The Mode II weight function $\mathbf{h}_2$ can be obtained similarly except that the Mode II modified singular displacement $\tilde{\mathbf{u}}_2^s$ will be prescribed instead.

The finite element mesh employed for this non-symmetrical crack geometry is shown in Fig. 6. The two weight functions were computed as outlined and the Mode I and II stress intensity factors, induced by uniform tensile loading σ applied on the top and bottom faces of the strip, were calculated. The normalized Mode I stress intensity factor, $K_1/(\sigma\sqrt{(\pi a)})$, is 2.169 and the corresponding normalized Mode II stress intensity factor, $K_2/(\sigma\sqrt{(\pi a)})$, is 0.4775. The results obtained from this coarse mesh compare well with those from Wilson[29].

5. CONCLUDING REMARKS

The present finite element method provides a unified approach for determining weight functions in two and three dimensions. The procedure outlined is simple and yet accurate and can be incorporated into any existing two- or three-dimensional displacement finite element programs with minimal effort. The procedure involves only prescribing the *surface loading* $-\tilde{\sigma}^s_\alpha \mathbf{n}^B$ on the inner surface S_{int} and an *equivalent body force* $\{B\}$ for those finite elements in region A that are next to the surface S_{int}. The determination of the weight function corresponding to each of the three fracture modes for symmetrical as well as non-symmetrical crack geometries can be obtained in a unified manner through the prescription of an appropriate modified singular displacement field. The formulation of the minimum principle allows the accurate determination of the modified regular displacements in the neighborhood of the crack tip. By combining these modified regular displacements with their corresponding modified singular displacements, we obtain a complete description of the weight functions in the entire body. This is a feature that is lacking in both the PMT method and the virtual crack extension approach. Another attractive feature of this method is that we only need to have a small collection of modified singular displacements of simple crack geometries in isotropic and anisotropic bodies in order to determine weight functions of complex crack geometries in finite bodies.

As shown in Section 2, in employing the PMT method, the core of material to be removed from the crack tip has to be very small in order that the approximation of $\tilde{\mathbf{u}}^r_\alpha(\mathbf{x}; Q)$ being zero is valid. This poses numerical difficulties because the energy of the functional being minimized increases like $1/d_0$ in the two-dimensional case and $1/d_0^2$ in the three-dimensional case as the characteristic size of the removed material, d_0, decreases. Such constraint on the size of region B in the PMT method is eliminated in the present method by *regularizing* the problem through an appropriately formulated minimum principle.

In the computation of three-dimensional weight functions[25], the present approach affords the calculation of weight functions either pointwisely along the crack front or as an integrated average by prescribing appropriate modified singular displacements $\tilde{\mathbf{u}}^s_\alpha$. Such added versatility allows the three-dimensional weight functions to be computed according to the degrees of approximation desired in engineering applications while retaining the option of generating weight functions pointwisely. In contrast, the three-dimensional computation procedure for weight functions proposed by Parks and Kamenetzky[16] only allows weighted averages of the stress intensity factors around the crack front to be obtained.

The applications of weight functions are not limited to the computations of the stress intensity factors of some cracked bodies under different applied loadings. Other applications of the two- and three-dimensional weight function theory are discussed by Rice[12, 13] and the finite element method introduced in this paper could be used to explore various facets of the weight function theory. Finally, it is remarked that the variational principle introduced in Appendix A is not limited to the determination of weight functions in the context of crack mechanics *per se*. The principle can be used to determine other *singular fields*, such as weight functions for notched bodies[6] and elastic Green's functions for finite bodies. It can also be employed to determine *regular fields* as noted in Appendix A.

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APPENDIX A

A variational principle for determining singular fields

In this appendix, we shall develop a general variational principle for determining elastic fields in finite bodies with unbounded strain energy. To this end we first distinguish two kinds of elastic fields; namely, *regular* fields and *singular* fields. Regular fields are fields of displacements, strains and stresses which produce finite elastic energy of deformation in any sub-domain of the body. They are the elastic response of imposed boundary displacements and continuously distributed boundary tractions and body forces. The displacements of a regular field have sufficient smoothness such that the divergence theorem applies. Singular fields are elastic fields which generate infinite elastic energy of deformation in the neighborhood of a singular point. Examples of singular fields are solutions to boundary value problems of point forces, dipoles, centers of compression and fundamental fields in cracked bodies.

To fix ideas, let us consider a homogeneous elastic body of finite extent, having arbitrary anisotropy. The body has volume V and bounding surface ∂V. The surface ∂V consists of ∂V_T (on which tractions are prescribed) and ∂V_u (on which displacements are imposed). Let Q be a singular point which lies in a closed sub-domain, B,

of the volume V. The bounding surface of B is ∂B and its unit outward normal is $\mathbf{n}^B$. The complement of the sub-domain B in V is A. Sub-domain A has a bounding surface ∂A and $\mathbf{n}^A$ is its outward unit normal. We label the various surfaces corresponding to the partition of V as follows:

$$\partial A_T = \partial A \cap \partial V_T \qquad \partial A_u = \partial A \cap \partial V_u$$
$$\partial B_T = \partial B \cap \partial V_T \qquad \partial B_u = \partial B \cap \partial V_u$$
$$S_{\text{int}} = \partial A \cap \partial B.$$

Without loss of generality, we assume that the singular point Q lies either on the boundary ∂B_T or in the interior of B. The unknown singular field has displacement $\mathbf{u}$, stress field $\boldsymbol{\sigma}$ and body force $\mathbf{f}$ in $A \cup B$, prescribed boundary traction $\mathbf{T}$ on $\partial A_T \cup \partial B_T$, and prescribed boundary displacements on $\partial A_u \cup \partial B_u$.

In sub-domain B, we approximate $\mathbf{u}$ by a *suitable* singular displacement $\mathbf{u}^s$ which has the same singularity as $\mathbf{u}$ at the singular point Q. The singular displacement $\mathbf{u}^s$ gives rise to a singular stress field $\boldsymbol{\sigma}^s$ which equilibrates a body force field $\mathbf{f}^s$ in B and exhibits traction $\boldsymbol{\sigma}^s\mathbf{n}^B$ on ∂B. The approximation $\mathbf{u}^s$ is chosen such that the difference $\mathbf{u}^r \equiv \mathbf{u} - \mathbf{u}^s$ represents the displacements of a regular field in B. The field exhibits the body force $\mathbf{f}^r \equiv \mathbf{f} - \mathbf{f}^s$ in B and the traction $\mathbf{T}^r \equiv \mathbf{T} - \boldsymbol{\sigma}^s\mathbf{n}^B$ on ∂B_T.

The displacements $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}^r$ are kinematically admissible in A and B respectively if:

(i) they satisfy the strain–displacement relations in their respective sub-domains;
(ii) $\bar{\mathbf{u}} - \bar{\mathbf{u}}^r = \mathbf{u}^s$ on S_{int};
(iii) and $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}^r + \mathbf{u}^s$ satisfy the geometric boundary conditions on ∂A_u and ∂B_u respectively.

Thus, with a proper choice for a singular field which satisfies all the conditions delineated above, the kinematically admissible fields $\bar{\mathbf{u}}$ and $\bar{\mathbf{u}}^r$ are regular in A and B, respectively.

Define a functional H for the elastic body as

$$H[\bar{\mathbf{u}}, \bar{\mathbf{u}}^r] = \int_A w(\bar{\varepsilon})\,dV - \int_A \mathbf{f}\cdot\bar{\mathbf{u}}\,dV - \int_{\partial A_T} \mathbf{T}\cdot\bar{\mathbf{u}}\,dA + \int_B w(\bar{\varepsilon}^r)\,dV - \int_B \mathbf{f}^r\cdot\bar{\mathbf{u}}^r\,dV - \int_{\partial B_T} \mathbf{T}^r\cdot\bar{\mathbf{u}}^r\,dA - \int_{S_{\text{int}}} (-\boldsymbol{\sigma}^s\mathbf{n}^B)\cdot\bar{\mathbf{u}}^r\,dA \quad \text{(A1)}$$

where w is the linear elastic strain energy density. The functional H is bounded and it is a functional of $\bar{\mathbf{u}}$ in sub-domain A and $\bar{\mathbf{u}}^r$ in sub-domain B. With these preliminaries established, a minimum principle for determining $\mathbf{u}$ in A and $\mathbf{u}^r$ in B can now be stated.

For all kinematically admissible fields $\bar{\mathbf{u}}$ in A and $\bar{\mathbf{u}}^r$ in B, the true fields $\mathbf{u}$ in A and $\mathbf{u}^r$ in B render the functional H a minimum. The proof of this minimum principle is as follows.

We introduce a difference state, denoted symbolically by Δ(state), as the difference between the kinematically admissible state and the true state. Then it follows that the elastic strain energy of the difference state in sub-domains A and B can be written respectively as

$$\int_A w(\Delta\varepsilon)\,dV = \int_A w(\bar{\varepsilon})\,dV - \int_A w(\varepsilon)\,dV - \int_A \boldsymbol{\sigma}\cdot\Delta\boldsymbol{\varepsilon}\,dV \quad \text{(A2)}$$

and

$$\int_B w(\Delta\varepsilon^r)\,dV = \int_B w(\bar{\varepsilon}^r)\,dV - \int_B w(\varepsilon^r)\,dV - \int_B \boldsymbol{\sigma}^r\cdot\Delta\boldsymbol{\varepsilon}^r\,dV. \quad \text{(A3)}$$

Thus, the difference in the two functionals corresponding to the kinematically admissible state and the true state is

$$\begin{aligned}\Delta H &= H[\bar{\mathbf{u}}, \bar{\mathbf{u}}^r] - H[\mathbf{u}, \mathbf{u}^r] \\ &= \int_A w(\Delta\varepsilon)\,dV + \int_A \boldsymbol{\sigma}\cdot\Delta\boldsymbol{\varepsilon}\,dV - \int_A \mathbf{f}\cdot\Delta\mathbf{u}\,dV - \int_{\partial A_T} \mathbf{T}\cdot\Delta\mathbf{u}\,dA \\ &\quad + \int_B w(\Delta\varepsilon^r)\,dV + \int_B \boldsymbol{\sigma}^r\cdot\Delta\boldsymbol{\varepsilon}^r\,dV - \int_B \mathbf{f}^r\cdot\Delta\mathbf{u}^r\,dV \\ &\quad - \int_{\partial B_T} \mathbf{T}^r\cdot\Delta\mathbf{u}^r\,dA - \int_{S_{\text{int}}} (-\boldsymbol{\sigma}^s\mathbf{n}^B)\cdot\Delta\mathbf{u}^r\,dA. \end{aligned} \quad \text{(A4)}$$

Now, the mixed energy in sub-domains A and B can be reduced respectively as

$$\int_A \boldsymbol{\sigma}\cdot\Delta\boldsymbol{\varepsilon}\,dV = \int_{\partial A_T + \partial A_u + S_{\text{int}}} (\boldsymbol{\sigma}\mathbf{n}^A)\cdot\Delta\mathbf{u}\,dA - \int_A (\operatorname{div}\boldsymbol{\sigma})\cdot\Delta\mathbf{u}\,dV \quad \text{(A5)}$$

$$\int_B \boldsymbol{\sigma}^r\cdot\Delta\boldsymbol{\varepsilon}^r\,dv = \int_{\partial B_T + \partial B_u + S_{\text{int}}} (\boldsymbol{\sigma}^r\mathbf{n}^B)\cdot\Delta\mathbf{u}^r\,dA - \int_B (\operatorname{div}\boldsymbol{\sigma}^r)\cdot\Delta\mathbf{u}^r\,dV. \quad \text{(A6)}$$

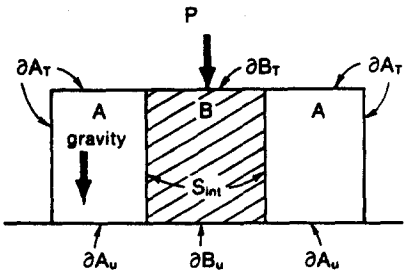

Fig. A1. The partitioning of a disk into sub-domains A and B. The disk is attached to a rigid foundation and is loaded by a point force P and gravity. The displacement $\mathbf{u}$ in B is approximated by the displacement $\mathbf{u}^s$ which corresponds to the Boussinesq solution of a point force P acting on a half-space.

We observe that the true field σ is in equilibrium with $\mathbf{f}$ in A and $\sigma\mathbf{n}^A = \mathbf{T}$ on ∂A_T; and similarly, σ^r equilibrates $\mathbf{f}^r$ in B and $\sigma\mathbf{n}^B = (\sigma^s+\sigma^r)\mathbf{n}^B = \mathbf{T}$ on ∂B_T. Furthermore, we have $\sigma = \sigma^s+\sigma^r$ and $\mathbf{n}^B = -\mathbf{n}^A$ on S_{int}; $\Delta\mathbf{u} = \mathbf{0}$ on ∂A_u; and $\Delta\mathbf{u}^r = \mathbf{0}$ on ∂B_u. Thus, substituting eqns (A5) and (A6) into eqn (A4), we obtain

$$\Delta H = \int_A w(\Delta\varepsilon)\,dV + \int_B w(\Delta\varepsilon^r)\,dV + \int_{S_{int}} (\sigma\mathbf{n}^B)\cdot(\Delta\mathbf{u}^r - \Delta\mathbf{u})\,dA. \tag{A7}$$

But on S_{int}

$$\begin{aligned}\Delta\mathbf{u}^r - \Delta\mathbf{u} &= [\bar{\mathbf{u}}^r - \bar{\mathbf{u}}] - [\mathbf{u}^r - \mathbf{u}]\\ &= \mathbf{u}^s - \mathbf{u}^s = 0\end{aligned}$$

thus we have

$$\bar{H} - H \geqslant 0$$

by the positive definiteness of the elasticity tensor associated with the strain energy density w and this proves the minimum principle.

A necessary condition for H to attain a minimum at $\bar{\mathbf{u}} = \mathbf{u}$ in A and $\bar{\mathbf{u}}^r = \mathbf{u}^r$ in B is

$$\delta H[\bar{\mathbf{u}}, \bar{\mathbf{u}}^r] = 0 \tag{A8}$$

where δH is the first variation of H. This variational statement (A8) gives rise to the following Euler–Lagrange equations and natural boundary conditions for the true state:

$$\begin{aligned}\operatorname{div}\sigma + \mathbf{f} &= \mathbf{0} && \text{in } A\\ \operatorname{div}\sigma^r + \mathbf{f}^r &= \mathbf{0} && \text{in } B\\ \sigma\mathbf{n}^A &= \mathbf{T} && \text{on } \partial A_T\\ (\sigma^r+\sigma^s)\mathbf{n}^B &= T && \text{on } \partial B_T\\ \sigma &= \sigma^s+\sigma^r && \text{on } S_{int}.\end{aligned}$$

To illustrate how this variational principle can be utilized, we consider the example of a disk attached to a rigid foundation, loaded by a point force P and the gravity. The disk can be divided into sub-domains A and B as depicted in Fig. A1. The applied traction $\mathbf{T}$ in this case is concentrated into the point force P, the body force $\mathbf{f}$ is the gravity and the displacements of the part of the disk which is attached to the rigid foundation are zero. We can take the Boussinesq solution of a point force P acting on a half-space as the approximation $\mathbf{u}^s$. With such a choice, the variational principle can be used as a basis for any suitable numerical procedures, such as Galerkin method or finite element method, to determine the displacements $\mathbf{u}$ in A and $\mathbf{u}^r$ in B.

The variational principle developed in this paper for determining singular fields in finite bodies with unbounded strain energy is a very general principle. As shown above, it forms a basis for numerical procedures which determine classical singular fields such as point force solutions in finite bodies. In Section 2, we show how this variational principle can be used to determine weight functions in cracked solids. Furthermore, the variational principle as it stands can also be employed to determine an unknown regular field in a finite body if a good approximation to the unknown regular field in a certain sub-domain of the body is available. We shall not pursue this aspect of the variational principle further in the present work.

APPENDIX B

Singular fields for a semi-infinite crack in two dimensions

Consider a semi-infinite crack in an infinite body which is linear elastic, isotropic and homogeneous. Under plane deformation, the displacement field $\mathbf{u}$ and its corresponding stress field can be expressed in terms of two analytic functions of z, $\phi(z)$ and $\rho(z)$, as[4]

$$2\mu(u_1+iu_2) = \kappa\phi(z)-\overline{\rho(z)}+(\bar{z}-z)\overline{\phi'(z)}$$
$$\sigma_{11} = \text{Re}\,[\phi'(z)+2\overline{\phi'(z)}-\overline{\rho'(z)}+(\bar{z}-z)\overline{\phi''(z)}]$$
$$\sigma_{22} = \text{Re}\,z(\phi'(z)+\overline{\rho'(z)}-(\bar{z}-z)\overline{\phi''(z)}]$$
$$\sigma_{12} = -\text{Im}\,[\phi'(z)-\rho'(z)+(z-\bar{z})\phi''(z)]$$

with

$$\kappa = 3-4\nu \qquad \text{(plane strain)}$$
$$\kappa = \frac{3-\nu}{1+\nu} \qquad \text{(generalized plane stress)}$$

where μ is the shear modulus, ν is Poisson's ratio, $z = x_1+ix_2$ is the complex variable, and i is the imaginary unit.

The singular displacements $\mathbf{u}_1^s$ and $\mathbf{u}_2^s$ of the semi-infinite crack and their corresponding stress fields $\boldsymbol{\sigma}_1^s$ and $\boldsymbol{\sigma}_2^s$ can be obtained by substituting in turn the following analytic functions into the above formulae

$$\phi_1(z) = \frac{-2\mu}{\kappa+1}\frac{\sqrt{(2\pi)}}{\sqrt{z}} \qquad \rho_1(z) = \frac{-2\mu}{\kappa+1}\frac{\sqrt{(2\pi)}}{\sqrt{z}} \qquad \text{for Mode I}$$

$$\phi_2(z) = \frac{2\mu i}{\kappa+1}\frac{\sqrt{(2\pi)}}{\sqrt{z}} \qquad \rho_2(z) = \frac{-2\mu i}{\kappa+1}\frac{\sqrt{(2\pi)}}{\sqrt{z}} \qquad \text{for Mode II.}$$

Under anti-plane strain deformation, the displacement and stresses are given by one analytic function of z, $\omega(z)$, as

$$\mu u_3 = \text{Im}\,[\omega(z)]$$
$$\sigma_{32}+i\sigma_{31} = \omega'(z).$$

The singular displacement $\mathbf{u}_3^s$ of a semi-infinite crack and its associated stress field in Mode III can be obtained by using the following form of $\omega(z)$[15]

$$\omega(z) = \frac{-\mu}{\sqrt{(2\pi z)}} \qquad \text{for Mode III.}$$